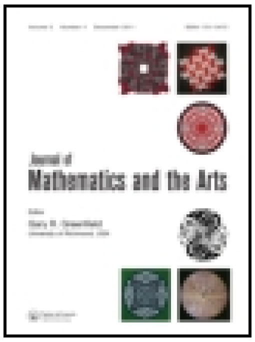

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Folding helical triangle tessellations into light art

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ABSTRACT
This article concerns the artistic and perceptual quality of translucent light transmitted by an origami-inspired paper surface when a light source is placed behind it. It describes my geometric strategies in origami design to create light art through the luminous effect of gradations of light. I first present some historical background and related work on origami-inspired paper light art and origami tessellation designs. Case studies follow, focusing on geometric strategies for helical triangle tessellations, considering specific design requirements for creating functional folded light art.

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Origami; tessellation; crease pattern; parametric; algorithm

1. Introduction

Light is non-material, but its presence has a strong effect on the people who perceive it. Light can be actively measured, but no amount of measurement can transmit the subjective quality and the phenomenon of how it is perceived. That largely depends on the type of surface material that interacts with the light source. This article describes how I, as an artist/designer, used geometric strategies in origami design to create light art through the luminous effect of the gradation of light. I first give some historical background and discuss related works on origami-inspired paper light art and origami tessellation designs. Case studies will illustrate geometric strategies found in helical triangle tessellations that must satisfy three specific design requirements for creating functional folded light art.

The first requirement is that the light art must be able to fold flat for shipping and storage and it must be able to deploy into a three-dimensional form. The second is that the folded surface must be seamlessly joined to create a continuous body to enclose the light source. The third is that there should be enough contrast in the illuminated origami form to bring out the best perceptual quality of a folded design. An artist can use mathematical strategies in origami design to create folded light art that satisfies all these conditions.

While a variety of materials can be used for folded light art (Figure 1), including both natural materials such as cotton paper and synthetic materials such as polypropylene and high-density polyethylene, the main material that is used for the light art presented here is Hi-tec Kozo paper, which is favoured for its beautiful natural warmth and its ability to scatter and transmit light to create a translucent glow. Hi-tec Kozo paper is a type of tear-free Shoji paper that has a three-layer structure, with eco-friendly polyester film as its core

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Figure 1. A collection of folded light art.

and Kozo Washi on both sides. In addition, stainless plates are laser-cut to specific shapes and dimensions to serve as luminary hardware, plastic snap buttons are used for connecting paper pieces, and warm LED globes that would distribute the light in a spherical form are used to evenly illuminate the light art from within.

2. Origami-inspired paper light art

There is a long history of paper light art that has been used both as a functional item for daily uses and as a symbolic item for ceremonial uses. Contemporary origami-inspired light, either folded from paper or from other foldable synthetic material, has become very popular, thanks to a few top-quality lighting designers and manufacturers in the world. Light passing through an origami paper surface creates beautiful translucent gradations of light, but also presents new challenges in its origami design.

2.1. Brief history of paper light

People of many different cultures have long preferred the soft diffused warm light transmitted by translucent materials such as paper and fabric instead of other dazzling light sources. In China, the tradition of making lanterns out of bamboo sticks and paper or silk goes back as early as 2000 years ago. Today, Chinese paper lights can be found as functional objects in homes and as decorative and symbolic items in festival activities such as the lantern festival. The Chinese lantern festival started in the East Han Dynasty when Emperor Ming ordered that lanterns be lit in order to honour the Buddhist spirit during the auspicious full moon period of each new lunar year [24]. Usually Chinese paper lights for homes are designed as a simple spherical or oblong plain form, while the paper lanterns for the festivals are decorated with vibrant colours and elaborate figures from myths in order to enhance the holiday spirit.

Japanese lanterns are similar to Chinese lanterns and are often made of paper or silk that is stretched over a bamboo stick or wire frame. Different types of paper or silk lanterns are used in different settings for various functions and have symbolic meaning. For example, Chouchin, often in an oblong shape, is used at the entrance of Buddhist temples, in traditional festivals, and at the entrances of bars and restaurants. Andon, often in a tetrahedral, cylindrical or cubic shape, is often used in the interiors of hotels and restaurants. Unlike Chouchin and Andon, Tourou is only used on special occasions, such as Toro Nagashi, the festival of the floating lantern [5]. The contemporary Akari lights made by Isamu Noguchi still use the traditional lantern-making techniques and have become a symbol of modern design worldwide [14].

Some areas of western culture also prefer the soft glow of paper or silk lanterns. Traditional luminaria, originating from Hispanic culture and often displayed during Christmas to kindle the spirit of Christ, is made from a paper bag with folded-down top and filled with a layer of sand that holds a lit candle. Danish people stay indoors during their dark and long winter months and have a preference for the soft glow of lights diffused by paper or other translucent material. Some of the world's best contemporary lighting equipment is made by Danish companies such as Le Klint and Louis Poulsen. Iconic Danish lights, such as the Artichoke Pendant light and I.Q. light, are very popular because of their combination of soft glow and intriguing geometries.

2.2. Origami-inspired lamp shades

While traditional lanterns made from paper or fabric require internal frames to support otherwise flimsy and insubstantial paper and fabric, some contemporary designs use origami-inspired folding techniques to create lamp shades without internal frames. Folding adds significant structural quality to material. Folds exist in many natural phenomena, from the earth's tectonic plates to graphene sheets. In man-made forms, folds can be found as early as in the pleated fabrics shown in Egyptian frescos and in ancient origami paper craft. The original purpose of origami was to create various shapes, ranging from animal figures to objects, and as decorative items to be used in religious and ceremonial activities. However, the craft techniques were also used to create functional objects. Perhaps one of the earliest functional models of origami was a folded gift box called Tamatebako, translated as 'magic treasure chest'. This first appeared in a Japanese book published in 1743 called Ranma Zushiki, which documented Edo-period design [12].

In the last few decades, constructing three-dimensional surfaces from two-dimensional sheet material has inspired artists, designers, architects and engineers to come up with folded sculptural forms that react to kinetic movements as well as the interplay of light and shadow in fashion, products and architecture. Danish architect Poul Christiansen designed a series of curve-folded lamps for Le Klint; these have been made since the 1970s. Today, Le Klint still produces their lamps by folding one piece of large paper or plastic. Japanese designer Issey Miyake is best known for his origami-inspired fashion designs that can be folded flat and expanded into three-dimensional forms to be worn. Recently, Miyake launched a collection of origami-inspired light sculptures called In-El Issey Miyake. In both origami-inspired Le Klint and In-El Issey Miyake, mathematical principles in both two and three dimensions were explored, resulting in sculptural forms that manipulated light and shadow in poetic ways.

2.3. Light and origami design

When light strikes the mountain and valley creases of a folded surface, it creates dramatic effects of gradations of light and shadow. But what are some perceptual qualities of origami light art and what are some mathematical perimeters an artist can use in origami design to bring out these perceptual qualities? When a light source is placed behind an origami structure that is folded from translucent material, the light does not pass directly through the material. It diffuses through the material, much like dye diffusing through a liquid. The glaring light source on the other side of the translucent material appears fuzzy and soft when seen from the outside. And it is precisely the perceptual quality of this warm fuzziness that has drawn people of different cultures since ancient times.

A light source positioned in front of an opaque shade will produce various lighting effects such as downlight, uplight, sidelight and backlight. However, when an origami-folded design of translucent material is lit from within, the positional relationship of the light source and the material can be essentially neglected; this has minimal effect on the perceptual quality of the light. Areas of the material that receive strong direct illumination tend to dissipate the light by transmitting it to other parts of the object. In order to create more dramatic effects with an origami light, the dihedral angles of the folds need to be carefully considered. The definition of the dihedral angle of an origami fold is different from the definition of a dihedral angle of a polyhedron. The *dihedral angle of an origami fold* is the angle between surface normals of the two incident sectors (faces). By contrast, the dihedral angle between two adjacent faces of a polyhedron is the plane angle between those faces. This means that the dihedral angle of an origami fold is the supplement of the angle in the polyhedron definition. When an origami fold is sharp, its dihedral angle is close to π.

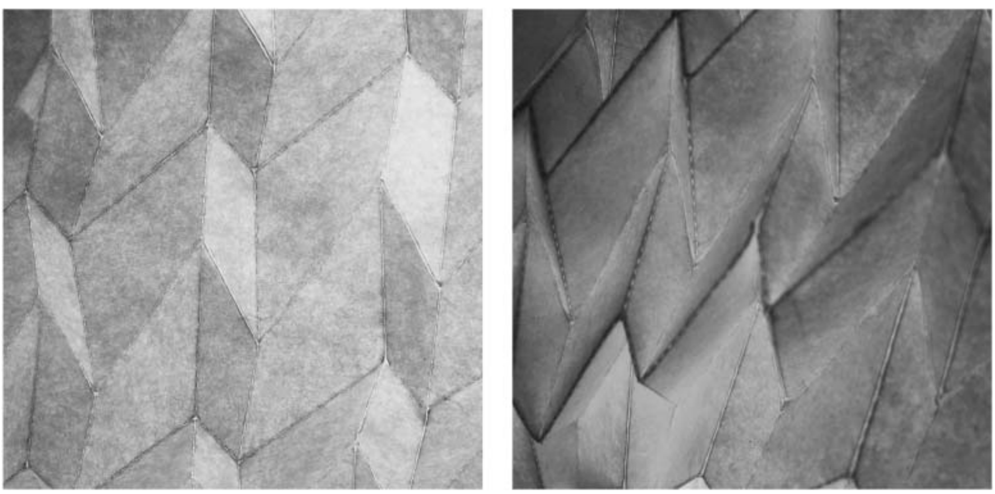


Figure 2. Comparison of an origami surface folded in paper, lit from behind. (a) Small dihedral angles of the folds reduce contrast on the illuminated surface. (b) Large dihedral angles of the folds increase contrast on the illuminated surface.

When the dihedral angles of the folds are small, the origami mountains and valleys seem more flat. Increasing the dihedral angles of the folds to make the folds sharper will bring out more dramatic gradation changes with more contrasts in illumination (Figure 2).

Another strategy to increase contrast on a folded origami form illuminated from behind is to vary the material's thickness. Less light will pass through the surface of a material that is double- or triple-layered, and the illuminated surface will appear to be darker in comparison to areas where there is only a single layer of material (Figure 3). Artist Chris Palmer has used this technique to create his beautiful illuminated *Shadow Folds* by creating origami tessellations on translucent textiles [23].

3. Origami and mathematics

Unfolding a piece of origami reveals the intricate crease patterns that define the geometrical transformation of folding the piece of material. The lines of the crease patterns will keep their length constant during the paper-folding transformation. Mathematicians call these geometric transformations isometric embeddings. This remarkable property found in paper folding suggests that there are deep connections between mathematics and origami. A new research field, origami mathematics, has been developed in the last decades in order to understand the mathematical formalization underlying paper folds. Origami mathematicians have studied many aspects of origami, such as folding polyhedra, tessellations and origami axioms. In 1997, mathematician Thomas Hull [9] created a modular origami model F.I.T. that was made of five intersecting tetrahedra, and today, there are many artists and mathematicians using modular origami techniques to create polyhedra [7]. Humiaki Huzita and Jacques Justin discovered seven axioms that are specific to origami. These axioms allow for certain geometric constructions not possible with classical Euclidean Axioms, including trisecting an arbitrary angle [10] and constructing the cubic root of integers [15].

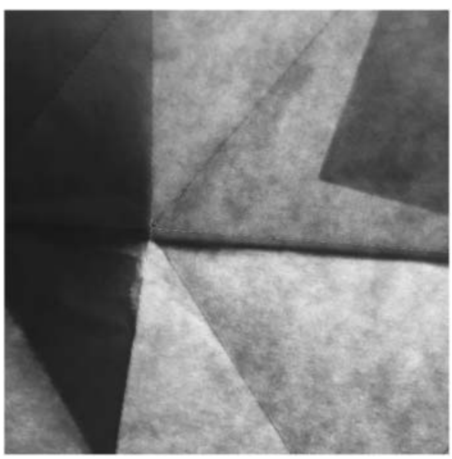


Figure 3. Illuminated layered paper showing dramatic contrast between areas that are triple-layered, double-layered and single-layered.

3.1. Origami tessellation

In geometry, tessellation refers to covering a flat surface by repeating a geometric shape, or multiple geometric shapes, with no overlaps or gaps. An origami tessellation is a folded design, where both the crease pattern and the folded structure use continuous and repeated elements. Origami tessellations were first developed by origami artists Yoshihide Momotani [17] and Shuzo Fujimoto [3], and their mathematical properties were first studied by Toshikazu Kawasaki and Masaaki Yoshida in the context of crystallographic symmetry [13]. In the 1960s and 1970s, computer scientist and artist Ronald Resch [22] and David Huffman [8] created many interesting tessellation designs. More recently, artist Chris Palmer [21] and Eric Gjerde [6] have pushed the art of origami tessellation to new heights. This new interest in origami tessellations have led artists, mathematicians and computer scientists within the origami community to look into how to create new origami tessellations by studying mathematical tiling [25] and computer algorithms [1]. For example, Alex Bateman's computer program Tess [2] allows users to create regular origami tessellations based on mathematical tilings such as Archimedean tilings.

3.2. Kawasaki's theorem and flat-foldable design

The crease pattern of an origami design refers to a set of mountain-folded lines (denoted by solid lines in this article) and valley-folded lines (denoted by dashed lines in this article) appearing when the folded structure is opened flat. A necessary and sufficient condition for a crease pattern to be flat-foldable locally is given by Kawasaki's theorem [16]:

Kawasaki's theorem. A crease pattern is flat-foldable locally if and only if

(I) *at each vertex (a point where crease lines meet), the number of lines meeting at that vertex is even, and*

(II) *the sum of alternating angles about that vertex is* 180°.

Figure 4(a) shows the crease pattern of an origami pinwheel design that is flat-foldable and its folded form. Figure 4(c) shows the crease pattern of an origami pinwheel design that does not satisfy condition (ii) of Kawasaki's theorem and therefore is not flat-foldable.

For an entire origami tessellation to be flat-foldable, the Kawasaki conditions must be satisfied by all the inner vertices of a crease pattern. In addition, there should be no

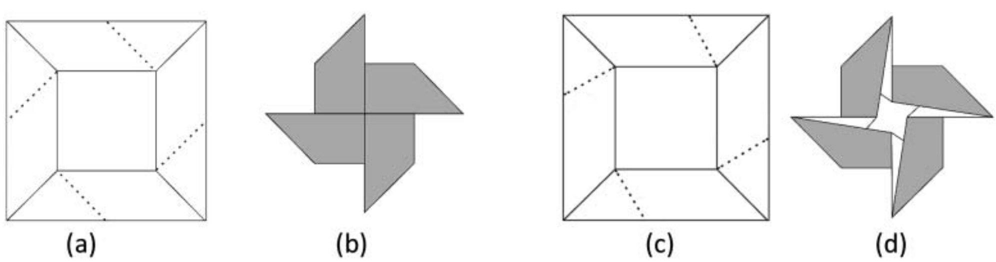


Figure 4. (a) Crease pattern of a flat-foldable origami pinwheel. (b) Origami pinwheel of (a). (c) Crease pattern of a non-flat-foldable origami pinwheel. (d) Origami pinwheel of (c).

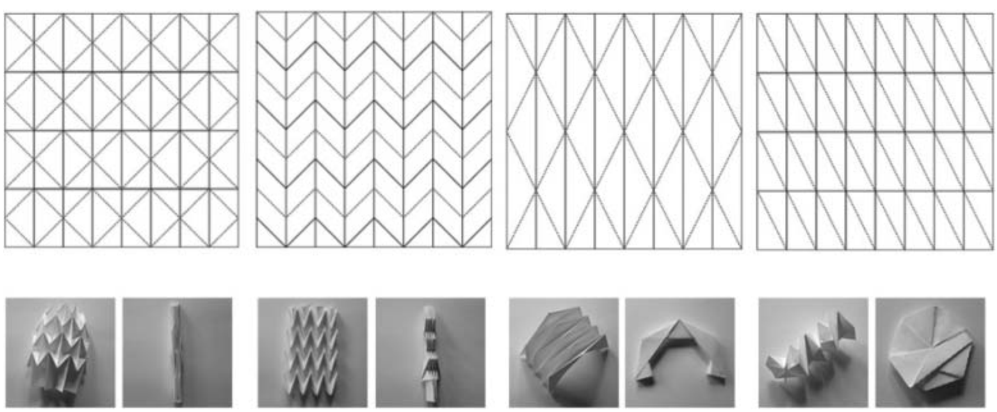
collision of the parts of the folded structure during assembly. Perhaps the most well-known flat-foldable and deployable origami tessellations are the Waterbomb pattern, the Miura pattern, the Yoshimura pattern and the Nojima helical triangle pattern. Figure 5 shows these four crease patterns and their folded forms in both their deployable and flat-folded states.

3.3. Helical triangle tessellation

The helical triangle tessellation (see Figure 5(d)) was first discussed extensively by Taketoshi Nojima, an engineer at Kyoto University. Nojima's helical triangle tessellation could be developed from his other spiral-like patterns that he called equiangular spirals in which a circular disc of paper was perfectly folded in the plane into a spiral-like design. In [18], Nojima gives a mathematical proof for creating flat-foldable equiangular spirals from a paper disc. In [19], he continued to demonstrate how helical triangle patterns (inspired by living organisms such as the phyllotaxies of leaves and flowers, insect wings, DNA and protein structures) could be folded in the axial direction. Nojima's helical patterns include periodic helical triangle patterns and their variations, as well as other helical patterns such as helical trapezoid patterns. One of the goals of Nojima's study was to use these foldable and deployable models to interpret the mechanics of unfolding flower buds and insect wings for potential use in building aerospace structures.

Many other artists and designers have also been fascinated by helix-like or spiral-like folding designs. The British artist Richard Sweeny and the American artist Matt Shilan have both created stunning paper sculptures based on origami helical designs. In [4], Japanese artist Tomoko Fuse demonstrated a wide range of spirals and helices of origami design through diagrams and photographs, and gave some mathematical explanation. While many artists have explored designs connected to origami helices and spirals, several of them have arrived at their results independently.

A *periodic helical triangle tessellation* has a crease pattern of repetitive units of congruent triangles with valley creases moving in a diagonal direction (see Figure 6(a)). One



(a) (b) (c) (d)

Figure 5. Crease patterns and the associated folded forms in deployable states and flat-foldable states: (a) Waterbomb pattern, (b) Miura pattern, (c) Yoshimura pattern and (d) Nojima's helical triangle pattern.

repeated unit consists of one parallelogram and the two triangles it contains; the diagonal of the parallelogram is a common edge of each triangle. The number of repeated units in the horizontal direction is denoted by i, and j is the number of repeated units in the vertical direction. Geometric properties of parallelograms and their diagonals ensure that all inner vertices of a periodic helical triangle tessellation satisfy Kawasaki's theorem, so it is flat-foldable and deployable.

4. Case studies of paper-folded light art

Through case studies, I will describe mathematical strategies and design strategies for creating paper-folded light art that satisfy the three design requirements discussed in Section 1: the light art must be flat-foldable, it must provide a continuously enclosed body for the light source and there must be pleasing contrast in the illuminated origami design.

In all the case studies, the positional relationship between the light source and the material is intentionally neglected. A warm LED globe with 360° of beam angle is placed in the centre of the enclosure of each of the origami structures in order to allow the light to smoothly fill up the entire origami body.

4.1. Periodic helical triangle tessellation light art

Since a periodic helical triangle tessellation satisfies Kawasaki's theorem, a light art folded from this crease pattern will be flat-foldable. To fold this crease pattern into a seamless cylindrical form outlined by regular polygons with n sides, acute angle α in Figure 6 must satisfy condition (1) as follows (α is equal to half an exterior angle of a regular n-gon):

$$\alpha = \frac{180^\circ}{n} \tag{1}$$

Angle β in Figure 6 can be arbitrary, but must satisfy $\alpha + \beta < 90^\circ$. As β decreases, γ increases, and the cylindrical column encloses a wider space.

When $n = 6$, $\alpha = 30^\circ$. Figure 7(a) shows a special case of a periodic helical triangle tessellation in which $\alpha = \beta$, and $n = 6$. Here the parallelograms are rhombs, and the pattern can be folded into a hexagonal column. The triangles in a periodic helical triangle tessellation can also be mirrored (shown shaded in Figure 7(b)) and the pattern still satisfies

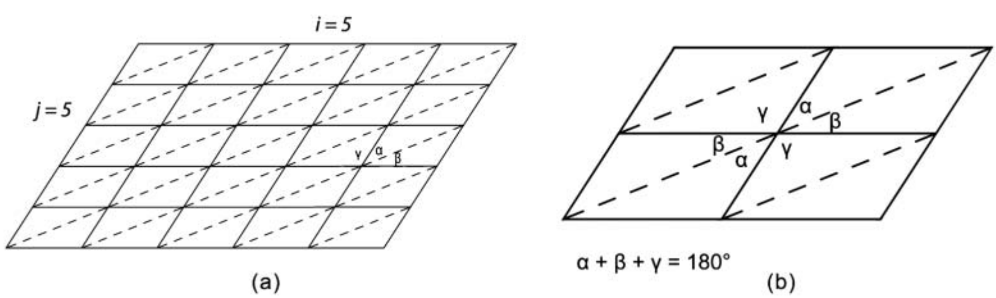


Figure 6. (a) A 5 × 5 periodic helical triangle crease pattern. (b) Each inner vertex from (a) satisfies Kawasaki's theorem.

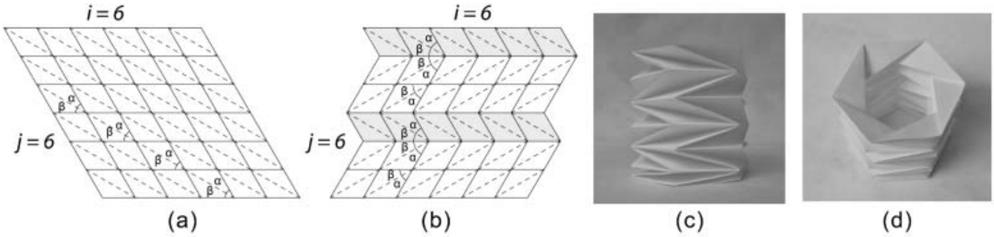


Figure 7. (a) A periodic helical triangle crease pattern. (b) A mirrored design derived from (a). (c) Pattern (b) in deployed state. (d) Pattern (b) in collapsed state.

Kawasaki's theorem, so is flat-foldable. This variation of Figure 7(a) produces a folded pattern that can be more interesting and easier to deploy. Figure 7(c,d) shows this variation in deployed state and collapsed into the form of a regular hexagon.

The last desired property for a folded light art is to increase the contrast of the illuminated origami form. The perceptual quality of contrast of a helical triangle light art is determined by the number of repeated units in the vertical direction of the tessellation. To design a light art of a fixed height at its deployed stage, an increase in j will increase the dihedral angles of the folds and therefore increase the contrast of illumination, but will be less efficient in terms of material usage. A decrease in j will result in the opposite effect. Figure 8 shows a light art with $j = 8$ that seems most proportional with appropriate contrasts of gradation in illumination.

4.2. Semi-periodic helical triangle light art

A semi-periodic helical triangle tessellation that can be folded into flat-foldable n-sided polygonal cylinder can be made by modifying the original periodic tessellation by giving β different values for two or more rows of units in the crease pattern. This modification does not change the fact that the pattern satisfies Kawasaki's theorem. Figure 9(a) shows an example of a semi-periodic flat-foldable helical triangle tessellation with mirrored

Figure 8. A lighted view of a periodic helical triangle light art.

triangles where $\alpha = 45°$ and angles β_1, β_2 and β_3 have different values. Figure 9(b,c) shows the resulting folded square column in its deployed state and its collapsed state.

Figure 10 shows a lighted view of a semi-periodic helical triangle light art. In order to bring out the best perceptual quality of the origami design, axial compression of the deployed light art is adjusted by experimenting with various β values and the number of repeated units in the vertical direction.

4.3. Self-similar helical triangle light art

A periodic helical triangle tessellation can also be developed into a self-similar helical triangle tessellation. Two figures are *self-similar* if corresponding lengths have the same ratio, that is, if one is either a magnification or a reduction of the other. The common ratio of corresponding lengths is referred to as the *scale factor*, *magnification* or *growth factor* [11].

A self-similar helical triangle tessellation (see Figure 11(c)) will fold into a cone. Though the tessellation I describe below was created using the Rhino CAD program, it could also be constructed using any dynamic geometry software, or by hand using a straight edge, compass and protractor. In Figure 11(a), the drawing begins with an isosceles triangle OAA′ with line AA′ as base and OA and OA′ the two equal sides. Let $\delta = \angle OAA'$ (δ is necessarily acute), and let μ be an arbitrary angle with $\mu < \delta$. Let α be another angle satisfying $\alpha < \delta - \mu$. The construction is as follows:

1. Rotate line AO clockwise about point A through angle μ to get line AM.
2. Rotate line AM clockwise about point A through angle α to get line AM′.
3. Rotate line A′O clockwise about point A′ through angle μ to get line AM″.
4. Line AM′ intersects line A′M″ at B′.
5. Draw an arc using point O as the centre and OB′ as radius.
6. The arc intersects line AM at point B.
7. Draw line BB′ (see Figure 11(a)).

Note that by construction, ΔAOB is congruent to ΔA′OB′, so ∠AOB = ∠A′OB′, which in turn implies ∠AOA′ = ∠BOB′. Thus ΔOBB′ is an isosceles triangle that is similar to ΔOAA′.

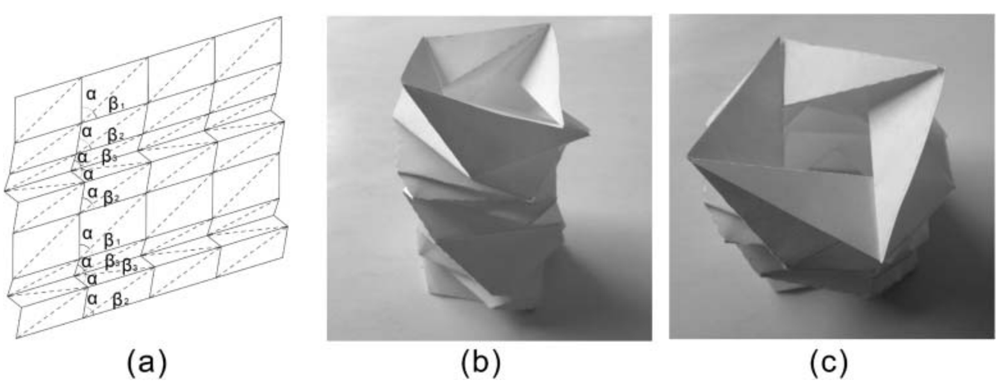


Figure 9. (a) A semi-periodic flat-foldable helical triangle tessellation. (b) The folded twisted square column in its deployed state and (c) in its collapsed state.

Figure 10. A lighted view of a semi-periodic helical triangle light art.

This means that the construction can be repeated beginning with ΔOBB′, and repeated again two more times to get Figure 11(b). Each of the quadrilaterals produced is similar to quadrilateral AA′B′B, with scale factor BB′/AA′. Also by the construction, a counter-clockwise rotation about O through angle AOA′ will map the spiral of quadrilaterals in Figure 11(b) to

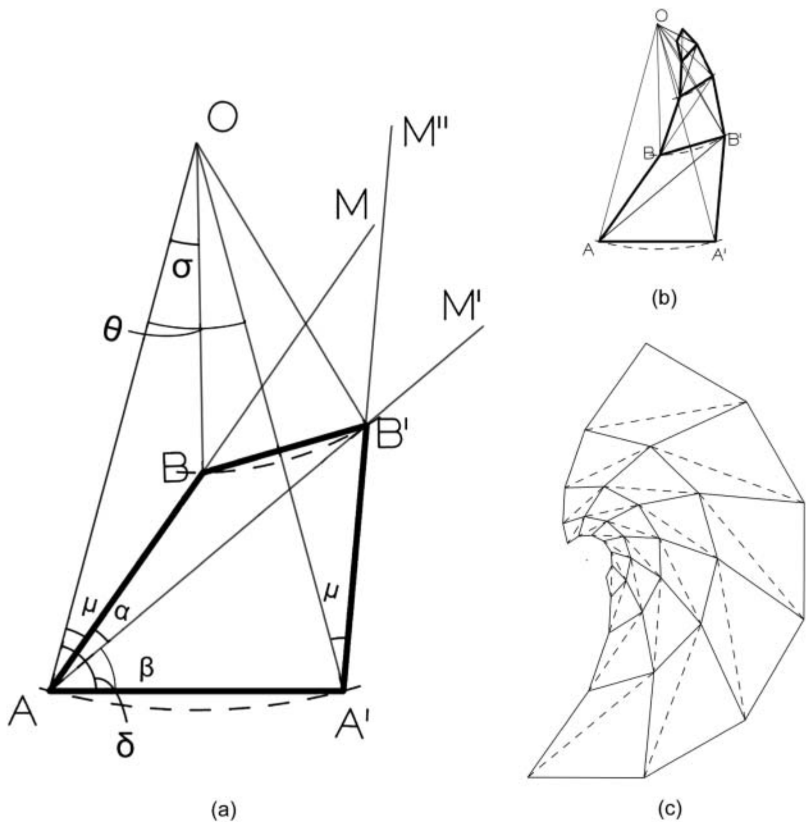


Figure 11. Geometric construction of a self-similar helical triangle tessellation.

an adjacent spiral. Repeating this rotation four more times produces the spiral crease pattern in Figure 11(c). Because of the self-similarity of all of the quadrilaterals in this pattern, each of the interior vertices of the crease pattern satisfies Kawasaki's theorem. Thus this crease pattern is flat-foldable, provided its interior vertices do not collide when folding.

While the geometric construction method described earlier is helpful when drawing a single self-similar helical triangle tessellation, parameterized algorithms can be used to construct self-similar tessellations dynamically, thus allowing the feasibility, proportion and variation of various folded designs to be tested quickly. In Figure 11(a), let $\sigma = \angle$AOB. Determining the value of σ is useful when drawing self-similar helical triangle tessellations using parametric algorithms. For a self-similar helical triangle tessellation to be folded into an n-edged polygonal cone, the angles δ and α in Figure 11(a) must satisfy $\delta > \frac{(n-2)}{n} \times 90^\circ$ and

$$\alpha = \delta - \frac{n-2}{n} \times 90^\circ \tag{2}$$

(Note that this angle α for the conical origami form is determined by the initial angle δ in the isosceles triangle AOA′, whereas the angle α in the helical origami form is determined by Equation (1).)

The scale factor r of the self-similar triangles in the construction is easily derived by using the law of sines on ΔAOB and the identity $\sin(180^\circ - x) = \sin(x)$:

$$\mathrm{r} = \mathrm{OB}/\mathrm{OA} = \frac{\sin(\mu)}{\sin(\mu + \sigma)} \tag{3}$$

To calculate the value of the angle σ in terms of the parameter n and chosen angles δ and μ, the law of sines and trigonometric identities are used for three triangles in Figure 11(a).

In ΔOAA′,

$$\frac{\mathrm{AA}'}{\mathrm{OA}} = \frac{\sin(2\delta)}{\sin(\delta)} = \frac{2\sin(\delta)\cos(\delta)}{\sin(\delta)} = 2\cos(\delta) \tag{i}$$

In ΔAOB,

$$\frac{\mathrm{OA}}{\mathrm{AB}} = \frac{\sin(\mu + \sigma)}{\sin(\sigma)} = \frac{\sin(\mu)\cos(\sigma) + \cos(\mu)\sin(\sigma)}{\sin(\sigma)}$$
$$= \sin(\mu)\cot(\sigma) + \cos(\mu) \tag{ii}$$

Let $\beta = \angle \mathrm{A'AB'}$. Then $\beta = \delta - \mu - \alpha = \frac{(n-2)\times 90^\circ}{n} - \mu$.

In ΔA′AB′,

$$\frac{\mathrm{AA}'}{\mathrm{A'B'}} = \frac{\sin(180^\circ - \delta - \mu - \beta)}{\sin(\beta)} = \frac{\sin\left(\frac{(n-2)\times 90^\circ}{n} + \delta\right)}{\sin\left(\frac{(n-2)\times 90^\circ}{n} - \mu\right)} \tag{iii}$$

Since AB = A′B′, $\frac{\mathrm{AA}'}{\mathrm{AB}} = \frac{\mathrm{AA}'}{\mathrm{OA}} \times \frac{\mathrm{OA}}{\mathrm{AB}} = \frac{\mathrm{AA}'}{\mathrm{A'B'}}$

Substituting from Equations (i), (ii) and (iii) gives

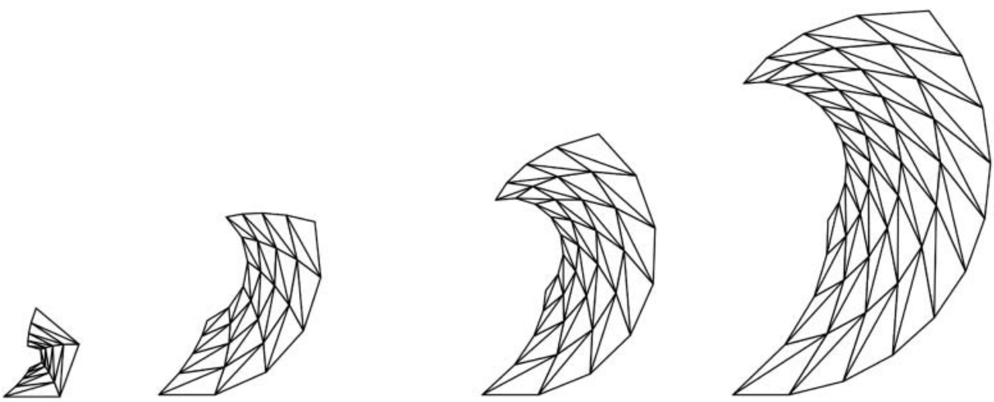
$$2\cos(\delta)(\sin(\mu)\cot(\sigma) + \cos(\mu)) = \frac{\sin\left(\frac{(n-2)\times 90^\circ}{n} + \delta\right)}{\sin\left(\frac{(n-2)\times 90^\circ}{n} - \mu\right)} \quad \text{(iv)}$$

Solving Equation (iv) for $\cot(\sigma)$ gives

$$\cot(\sigma) = \frac{\sin\left(\frac{(n-2)\times 90^\circ}{n} + \delta\right)}{2\sin(\mu)\cos(\delta)\sin\left(\frac{(n-2)\times 90^\circ}{n} - \mu\right)} - \cot(\mu) \quad (4)$$

In order to study the variations of self-similar helical triangle tessellations and their potential use for folding into lighted sculptural forms, a parametric algorithm was built in Rhino/Grasshopper in order to generate unlimited variations of these tessellations. The algorithm is based on a series of geometric construction steps, as well as Equations (3) and (4), which calculate the angle σ and growth factor r. By changing the parameters n (the number of edges of a regular polygon cone folded from the tessellation), μ and δ, the algorithm is able to output a variety of self-similar helical triangle tessellations (Figure 12). It is important to note that n, α and δ cannot be arbitrary. The relationship between n, α and δ needs further study.

Figure 13 shows a light art that is made by joining a hexagonal cone and its inverse version. In order to conceal the seams where the two cones meet, I changed the way the paper was cut. Instead of folding this from two pieces of paper, I folded it from six pieces of Hi-tec Kozo in order to get a cleaner and more consistent look at the seams. Each of the hexagonal cones is folded from a self-similar helical triangle pattern produced from the algorithm programme with $n = 6$, $\delta = 75^\circ$ and $\mu = 20^\circ$. In each of



$n=3$	$n=5$	$n=7$	$n=10$
$\delta=55$	$\delta=78$	$\delta=79$	$\delta=83$
$\mu=15$	$\mu=31$	$\mu=36$	$\mu=39$

Figure 12. Variations of self-similar helical triangle tessellations that can be folded into n-edged flat-foldable seamless polygonal cones.

Figure 13. Lighted view of self-similar helical light art with $n = 6$, $\delta = 75°$ and $\mu = 20°$.

the hexagonal cones, the number of repeated units in the vertical direction j is 4. As before, if the deployed height of the light art remains fixed, an increase in j will increase the contrast in illumination with more compression in the axial direction, but in this case will produce a design whose top opening is too small to insert a light bulb. By changing the parameters n, δ, μ, i and j, various self-similar helical triangle tessellations can be quickly generated and folded to study their proportions and the contrast of illumination for creating dramatic lighting effects.

5. Conclusion

In this article, I have described how as an artist/designer I explored the relationships between geometric strategies in helical triangle tessellations and the perceptual quality of illuminated origami design in order to create light art with interesting effects of light gradation. Though helical triangle tessellations are credited to Taketoshi Nojima and have also been explored by origami artists such as Tomoko Fuse, the geometric construction methods and algorithms that are presented here are the result of my own artistic explorations. There are certainly many other aspects of illuminated origami design and perceptual light quality that deserve further study:

- How does the folded surface interfere with the scattering of light that is passing through it from behind?
- How does the folded surface change the effect of downlighting, uplighting, sidelighting and backlighting?

Answering these questions will help us design effective lighting for both private and public spaces. In addition, there needs to be a more advanced understanding of helical triangle tessellations. Further explorations of mathematical strategies in designing helical triangle tessellations may reveal more creative potential of these patterns.

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